Chapter 2

Complex Numbers

The purpose of this document is to give you a brief overview of complex numbers, notation associated with complex numbers, and some of the basic operations involving complex numbers.

The complex number system Introduction

In this section we shall define the complex number system as the set $\mathbf{R} \times \mathbf{R}$ (the Cartesian product of the set of reals, $\mathbf{R}$, with itself) with suitable addition and multiplication operations. We shall define the real and imaginary parts of a complex number and compare the properties of the

complex number system with those of the real number system, particularly from the point of view of analysis.

Defining the complex number system

In complex analysis we are concerned with functions whose domains and codomains are subsets of the set of complex numbers. As you probably know, this structure is obtained from the set $\mathbf{R} \times \mathbf{R}$ by defining suitable operations of addition and multiplication. This reveals immediately one important difference between real analysis and complex analysis: in real analysis we are concerned with sets of real numbers, in complex analysis we are concerned with sets of **ordered pairs** of real numbers.

Whatever context is used to introduce complex numbers, one sooner or later meets the symbol i and the strange formula $i^2 = -1$.

Historically, the notion of a "number" i with this property arose from the desire to extend the real number system so that equations such as $\boldsymbol{x^2 + 1 = 0}$ have solutions. There is no **real** number satisfying this equation so, as usual in mathematics, it was decided to **invent** a number system that did contain a

solution. The remarkable fact is that having invented a solution of this **one** equation, we can use it to construct a system that contains the solutions of **every** polynomial equation!. For example, using the well-known formulas

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a}$$

for the solutions of the equation $ax^2+bx+c=0$, where $a \neq 0$,

Example: we find that the solutions of $x^2+x+1=0$ are apparently given by the expressions

$$\frac{-1+\sqrt{-3}}{2}, \text{ and } \frac{-1-\sqrt{-3}}{2}.$$

These make no sense at all until we turn a blind eye to $\sqrt{-3}$ and just manipulate it formally, as though we knew what we were doing, to give

$$\sqrt{-3} = \sqrt{3\times -1} = \sqrt{3}\sqrt{-1}.$$

We then say that if $i^2 = -1$ then we might as well press on and replace $\sqrt{-1}$ by *i* and so the "solutions" of the equation are

$$\frac{-1}{2} + \frac{i\sqrt{3}}{2}, \text{ and } \frac{-1}{2} - \frac{i\sqrt{3}}{2}.$$

A new number called "*i*", standing for "imaginary",. (That's why you couldn't take the square root of a negative number before: you only had "real" numbers; that is, numbers without the "*i*" in them.) The imaginary is defined to be:

$i = \sqrt{-1}$ Then $i^2 = \left(\sqrt{-1}\right)^2 = -1$

Now, you may think you can do this:

$$i^2 = \left(\sqrt{-1}\right)^2 = \sqrt{(-1)^2} = \sqrt{1} = 1$$

This points out an important detail: When dealing with imaginaries, you gain something (the ability to deal with negatives inside square roots), but you also lose something (some of the flexibility and convenient rules you used to have when dealing with square roots). In particular, YOU MUST ALWAYS DO THE *i*-PART FIRST!

Simplify *sqrt* (–9).

$$\sqrt{-9} = \sqrt{9 \cdot (-1)} = \sqrt{9}\sqrt{-1} = \sqrt{9} \cdot i = 3i$$

(Warning: The step that goes through the third "equals" sign is " $\sqrt{-1} = i$ ", not "$\sqrt{-1} = \sqrt{i}$ ".)

Simplify *sqrt*(–25).

$$\sqrt{-25} = \sqrt{25 \cdot (-1)} = \sqrt{25}\sqrt{-1} = 5i$$

Simplify *sqrt*(–18).

$$\sqrt{-18} = \sqrt{9 \cdot 2 \cdot (-1)} = \sqrt{9}\sqrt{2}\sqrt{-1} = 3\sqrt{2}i$$

Simplify –*sqrt*(–6).

$-\sqrt{-6} = -\sqrt{6 \cdot (-1)} = -\sqrt{6}\sqrt{-1} = -\sqrt{6}i$

In your computations, you will deal with *we* just as you would with *x*, except for the fact that x^2 is just x^2, but i^2 is –1:

Simplify 2*i* + 3*i*.

$2i + 3i = (2 + 3)i =$ $\mathbf{5}\boldsymbol{i}$

Simplify 16*i* – 5*i*.

$16i - 5i = (16 - 5)i =$ $\mathbf{11}\boldsymbol{i}$

Multiply and simplify (3*i*)(4*i*).

$(3i)(4i) = (3 \cdot 4)(i \cdot i) = (12)(i^2) = (12)(-1) =$ **–12**

Multiply and simplify (*i*)(2*i*)(–3*i*).

$(i)(2i)(-3i) = (2 \cdot -3)(i \cdot i \cdot i) = (-6)(i^2 \cdot i)$

$=(-6)(-1 \cdot i) = (-6)(-i) = \mathbf{6}\boldsymbol{i}$

The Definition

As I've already stated, I am assuming that you're aware that $i = \sqrt{-1}$ and so $i^2 = -1$. This is an idea that most people first see in an algebra class (or wherever they first saw complex numbers) and $i = \sqrt{-1}$ is defined so that we can deal with square roots of negative numbers as follows,

$$\sqrt{-100} = \sqrt{(100)(-1)} = \sqrt{100}\,\sqrt{-1} = \sqrt{100}\, i = 10i$$

What I'd like to do is give a more *mathematical* definition of a complex numbers and show that $i^2 = -1$ (and hence $i = \sqrt{-1}$) can be thought of as a consequence of this definition. We'll also take a look at how we define arithmetic for complex numbers.

What we're going to do here is going to seem a little backwards from what you've probably already seen but is in fact a more accurate and mathematical definition of complex numbers. Also note that this section is not really required to understand the remaining portions of this document. It is here solely to show you a different way to define complex numbers. So, let's give the definition of a complex number.

Given two real numbers a and b we will define the complex number z as,

$$z = a + bi \quad (1)$$

Note that at this point we've not actually defined just what i is at this point. The number $\boldsymbol{a}$ is called the **real part** of $\boldsymbol{z}$ and the number $\boldsymbol{b}$ is called the **imaginary part** of z and are often denoted as,

$$\operatorname{Re} z = a \qquad\qquad \operatorname{Im} z = b \quad (2)$$

There are a couple of special cases that we need to look at before proceeding. First, let's take a look at a complex number that has a zero real part,

$$z = 0 + bi = bi$$

In these cases, we call the complex number a **pure imaginary** number.

Next, let's take a look at a complex number that has a zero imaginary part,

$$z = a + 0i = a$$

In this case we can see that the complex number is in fact **a real number.** Because of this we can think of the real numbers as being a subset of the complex numbers.

We next need to define how we do addition and multiplication with complex numbers. Given two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ we define addition and multiplication as follows,

$$z_1 + z_2 = (a+c) + (b+d)i \qquad (3)$$

$$z_1 z_2 = (ac - bd) + (ad + cb)i \qquad (4)$$

Now, if you've seen complex numbers prior to this point you will probably recall that these are the formulas that were given for addition and multiplication of complex numbers at that point. However, the multiplication formula that you were given at that point in time required the use of $i^2 = -1$ to completely derive and for this section we don't yet know that is true. In fact, as noted previously $i^2 = -1$ will be a consequence of this definition as we'll see shortly.

Above we noted that we can think of the real numbers as a subset of the complex numbers. Note that the formulas for addition and multiplication of complex numbers give the standard real number formulas as well. For instance given the two complex numbers,

$z_1 = a + 0i \qquad\qquad z_2 = c + 0i$

the formulas yield the correct formulas for real numbers as seen below.

$$z_1 + z_2 = (a+c) + (0+0)i = a + c$$

$$z_1 z_2 = (ac - (0)(0)) + (a(0) + c(0))i = ac$$

The last thing to do in this section is to show that $i^2 = -1$ is a consequence of the definition of multiplication. However, before we do that we need to acknowledge that powers of complex numbers work just as they do for real numbers. In other words, if n is a positive integer we will define exponentiation as,

$$z^n = \underbrace{z \cdot z \cdot \cdots \cdot z}_{n \text{ times}}$$

Complex Arithmetic

Example 1 Compute each of the following.

(a) $(58 - i) + (2 - 17i)$ **(b)** $(6 + 3i)(10 + 8i)$ **(c)** $(4 + 2i)(4 - 2i)$

Solution

As noted above, I'm assuming that this is a review for you and so won't be going into great detail here.

(a) $\left(58-i\right)+\left(2-17i\right)=58-i+2-17i=60-18i$

(b) $\left(6+3i\right)\left(10+8i\right)=60+48i+30i+24i^2=60+78i+24\left(-1\right)=36+78i$

(c) $\left(4+2i\right)\left(4-2i\right)=16-8i+8i-4i^2=16+4=20$

It is important to recall that sometimes when adding or multiplying two complex numbers the result might be a real number as shown in the third part of the previous example!

The third part of the previous example also gives a nice property about complex numbers.

$$\left(a+bi\right)\left(a-bi\right)=a^2+b^2 \qquad (1)$$

We'll be using this fact with division and looking at it in slightly more detail in the next section.

Let's now take a look at the subtraction and division of two complex numbers. Hopefully, you recall that if we have two complex numbers, $z_1 = a + bi$ and $z_2 = c + di$ then you subtract them as,

$$z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i \qquad (2)$$

And that division of two complex numbers,

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di} \qquad (3)$$

can be thought of as simply a process for eliminating the i from the denominator and writing the result as a new complex number $u + vi$.

Let's take a quick look at an example of both to remind us how they work.

Example 2 Compute each of the following.

(a) $(58-i)-(2-17i)$ **(b)** $\dfrac{6+3i}{10+8i}$ **(c)** $\dfrac{5i}{1-7i}$

Solution

(a) There really isn't too much to do here so here is the work,

$$(58-i)-(2-17i)=58-i-2+17i=56+16i$$

(b) Recall that with division we just need to eliminate the i from the denominator and using (1) we know how to do that. All we need to do is multiply the numerator and denominator by $10-8i$ and we will eliminate the i from the denominator.

$$
\begin{aligned}
\frac{6+3i}{10+8i} &= \frac{(6+3i)}{(10+8i)}\frac{(10-8i)}{(10-8i)} \\
&= \frac{60-48i+30i-24i^2}{100+64} \\
&= \frac{84-18i}{164} \\
&= \frac{84}{164}-\frac{18}{164}i = \frac{21}{41}-\frac{9}{82}i
\end{aligned}
$$

(c) We'll do this one a little quicker.

$$
\frac{5i}{1-7i} = \frac{5i}{(1-7i)}\frac{(1+7i)}{(1+7i)} = \frac{-35+5i}{1+49} = -\frac{7}{10}+\frac{1}{10}i
$$

Now, for the most part this is all that you need to know about subtraction and multiplication of complex numbers for this rest of this document. However, let's take a look at a more precise and mathematical definition of both of these. If you aren't interested in this then you can skip this and still be able to understand the remainder of this document.

Technically, the only arithmetic operations that are defined on complex numbers are addition and multiplication. This means that both subtraction and division will, in some way, need to be defined in terms of these two operations. We'll start with subtraction since it is (hopefully) a little easier to see.

We first need to define something called an **additive inverse**. An additive inverse is some element typically denoted by $-z$ so that

$$z+(-z)=0 \qquad (4)$$

so for a given complex number $z=a+bi$ the additive inverse, $-z$, is given by, $-z=(-1)z=-a-bi$

It is easy to see that this does meet the definition of the additive inverse and so that won't be shown.

With this definition we can now officially define the subtraction of two complex numbers.

Given two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ we define the subtraction of them as,

$$z_1 - z_2 = z_1 + (-z_2) \qquad (5)$$

Or, in other words, when subtracting z_2 from z_1 we are really just adding the additive inverse of z_2 (which is denoted by $-z_2$) to z_1. If we further use the definition of the additive inverses for complex numbers we can arrive at the formula given above for subtraction.

$$z_1 - z_2 = z_1 + (-z_2) = (a + bi) + (-c - di) = (a - c) + (b - d)i$$

It's just that in all of the examples where you are liable to run into the notation $-z$ in "real life", whatever that means, we really do mean $-z = (-1)z$.

As with subtraction we first need to define an inverse. This time we'll need a **multiplicative inverse**. A multiplicative inverse for a non-zero complex number z is an element denoted by z^{-1} such that

$$z\,z^{-1} = 1$$

Now, again, be careful not to make the assumption that the "exponent" of -1 on the notation is in fact an exponent. It isn't! It is just a notation that is used to denote the multiplicative inverse. With real (non-zero) numbers this turns out to be a real exponent and we do have that

$$4^{-1} = \frac{1}{4}$$

for instance. However, with complex numbers this will not be the case! In fact, let's see just what the multiplicative inverse for a complex number is. Let's start out with the complex number $z = a + bi$ and let's call its multiplicative inverse $z^{-1} = u + vi$. Now, we know that we must have

$$z z^{-1} = 1$$

so, let's actual do the multiplication.

$$\begin{aligned} zz^{-1} &= (a+bi)(u+vi) \\ &= (au-bv)+(av+bu)i \\ &= 1 \end{aligned}$$

This tells us that we have to have the following,

$$au - bv = 1 \qquad\qquad av + bu = 0$$

Solving this system of two equations for the two unknowns *u* and *v* (remember *a* and *b* are known quantities from the original complex number) gives,

$$u = \frac{a}{a^2+b^2} \qquad\qquad v = -\frac{b}{a^2+b^2}$$

Therefore, the multiplicative inverse of the complex number *z* is,

$$z^{-1} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}\, i \qquad (6)$$

As you can see, in this case, the "exponent" of -1 is not in fact an exponent! Again, you really need to *forget* some notation that you've become familiar with in other math courses.

we can finally define division of two complex numbers. Suppose that we have two complex numbers z_1 and z_2 then the division of these two is defined to be,

$$\frac{z_1}{z_2} = z_1 z_2^{-1} \tag{7}$$

In other words, division is defined to be the multiplication of the numerator and the multiplicative inverse of the denominator. Note as well that this actually does match with the process that we used above. Let's take another look at one of the examples that we looked at earlier only this time let's do it using multiplicative inverses. So, let's start out with the following division.

$$\frac{6+3i}{10+8i}=(6+3i)(10+8i)^{-1}$$

We now need the multiplicative inverse of the denominator and using (6) this is,

$$(10+8i)^{-1}=\frac{10}{10^2+8^2}-\frac{8}{10^2+8^2}i=\frac{10-8i}{164}$$

Now, we can do the multiplication,

$$\frac{6+3i}{10+8i}=(6+3i)(10+8i)^{-1}=(6+3i)\frac{10-8i}{164}=\frac{60-48i+30i-24i^2}{164}=\frac{21}{41}-\frac{9}{82}i$$

Notice that the second to last step is identical to one of the steps we had in the original working of this problem and, of course, the answer is the same.

As a final topic let's note that if we don't want to remember the formula for the multiplicative inverse we can get it by using the process we used in the original multiplication. In other words, to get the multiplicative inverse we can do the following

$$(10+8i)^{-1} = \frac{1}{(10+8i)}\frac{10-8i}{(10-8i)} = \frac{10-8i}{10^2+8^2}$$

As you can see this is essentially the process we used in doing the division initially.

Conjugate and Modulus

In the previous section we looked at algebraic operations on complex numbers. There are a couple of other operations that we should take a look at since they tend to show up on occasion. We'll also take a look at quite a few nice facts about these operations.

Complex Conjugate

The first one we'll look at is ***the complex conjugate***, (or just the conjugate). Given the complex number $z = a + bi$ the complex conjugate is denoted by $\bar{z}$ and is defined to be,

$$\bar{z} = a - bi \qquad (1)$$

In other words, we just switch the sign on the imaginary part of the number. Here are some basic facts about conjugates.

$$\bar{\bar{z}} = z \qquad (2)$$

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2 \qquad (3)$$

$$\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2 \qquad (4)$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \qquad (5)$$

The first one just says that if we conjugate twice we get back to what we started with originally and hopefully this makes some sense. The remaining three just say we can break up sum, differences, products and quotients into the individual pieces and then conjugate.

So, just so we can say that we worked a number example or two let's do a couple of examples illustrating the above facts.

Example 1 Compute each of the following.

(a) $\overline{\overline{z}}$ for $z = 3 - 15i$ **(b)** $\overline{z_1 - z_2}$ for $z_1 = 5 + i$ and $z_2 = -8 + 3i$

(c) $\overline{z}_1 - \overline{z}_2$ for $z_1 = 5 + i$ and $z_2 = -8 + 3i$

Solution

There really isn't much to do with these other than to so the work so,

(a) $\overline{z} = 3 + 15i \quad \Rightarrow \quad \overline{\overline{z}} = \overline{3 + 15i} = 3 - 15i = z$

Sure enough we can see that after conjugating twice we get back to our original number.

(b) $z_1 - z_2 = 13 - 2i \qquad \Rightarrow \qquad \overline{z_1 - z_2} = \overline{13 - 2i} = 13 + 2i$

(c) $\overline{z}_1 - \overline{z}_2 = \overline{5+i} - \left(\overline{-8+3i}\right) = 5 - i - (-8 - 3i) = 13 + 2i$

We can see that results from **(b)** and **(c)** are the same as the fact implied they would be.

There is another nice fact that uses conjugates that we should probably take a look at. However, instead of just giving the fact away let's derive it. We'll start with a complex number $z = a + bi$ and then perform each of the following operations.

$$\begin{aligned} z + \overline{z} &= a + bi + (a - bi) \\ &= 2a \end{aligned}$$

$$\begin{aligned} z - \overline{z} &= a + bi - (a - bi) \\ &= 2bi \end{aligned}$$

Now, recalling that $\operatorname{Re} z = a$ and $\operatorname{Im} z = b$ we see that we have,

$$\operatorname{Re} z = \frac{z+\bar{z}}{2} \qquad\qquad \operatorname{Im} z = \frac{z-\bar{z}}{2i} \qquad (6)$$

Modulus

The other operation we want to take a look at in this section is the **modulus** of a complex number. Given a complex number $z = a + bi$ the modulus is denoted by $|z|$ and is defined by

$$|z| = \sqrt{a^2 + b^2} \qquad (7)$$

Notice that the modulus of a complex number is always a real number and in fact it will never be negative since square roots always return a positive number or zero depending on what is under the radical.

Notice that if *z* is a real number (*i.e.* $z = a + 0i$) then, $|z| = \sqrt{a^2} = |a|$

where the $|\cdot|$ on the z is the modulus of the complex number and the $|\cdot|$ on the a is the absolute value of a real number (recall that in general for any real number a we have $\sqrt{a^2} = |a|$). So, from this we can see that for real numbers the modulus and absolute value are essentially the same thing.

We can get a nice fact about the relationship between the modulus of a complex numbers and its real and imaginary parts. To see this let's square both sides of (7) and use the fact that $\operatorname{Re} z = a$ and $\operatorname{Im} z = b$. Doing this we arrive at

$$|z|^2 = a^2 + b^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2$$

Since all three of these terms are positive we can drop the Im z part on the left which gives the following inequality,

$$|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \geq (\operatorname{Re} z)^2$$

If we then square root both sides of this we get, $\left|z\right| \ge \left|\operatorname{Re} z\right|$
where the $\left|\,\cdot\,\right|$ on the z is the modulus of the complex number and the $\left|\,\cdot\,\right|$ on the Re z are absolute value bars. Finally, for any real number a we also know that $a \le \left|a\right|$ (absolute value…) and so we get,

$$\left|z\right| \ge \left|\operatorname{Re} z\right| \ge \operatorname{Re} z \qquad (8)$$

We can use a similar argument to arrive at,

$$\left|z\right| \ge \left|\operatorname{Im} z\right| \ge \operatorname{Im} z \qquad (9)$$

There is a very nice relationship between the modulus of a complex number and it's conjugate. Let's start with a complex number $z = a + bi$ and take a look at the following product.

$$z\,\overline{z} = \left(a+bi\right)\left(a-bi\right) = a^2 + b^2$$

From this product we can see that

$$z\,\bar{z} = |z|^2 \qquad (10)$$

This is a nice and convenient fact on occasion.

Notice as well that in computing the modulus the sign on the real and imaginary part of the complex number won't affect the value of the modulus and so we can also see that,

$$|z| = |\bar{z}| \qquad (11) \qquad \text{and}$$

$$|-z| = |z| \qquad (12)$$

We can also now formalize the process for division from the previous section now that we have the modulus and conjugate notations. In order to get the *i* out of the denominator of the quotient we really multiplied the numerator and denominator by the conjugate of the denominator. Then using (10) we can simplify the notation a little. Doing all this gives the following formula for division,

$$\frac{z_1}{z_2} = \frac{z_1\,\overline{z}_2}{z_2\,\overline{z}_2} = \frac{z_1\,\overline{z}_2}{\left|z_2\right|^2}$$

Here's a quick example illustrating this,

Example 2 Evaluate $\dfrac{6+3i}{10+8i}$.

Solution

In this case we have $z_1 = 6+3i$ and $z_2 = 10+8i$. Then computing the various parts of the formula gives,

$$\overline{z}_2 = 10-8i \qquad\qquad \left|z_2\right|^2 = 10^2 + 8^2 = 164$$

The quotient is then,

$$\frac{6+3i}{10+8i} = \frac{(6+3i)(10-8i)}{164} = \frac{60-48i+30i-24i^2}{164} = \frac{21}{41} - \frac{9}{82}i$$

Here are some more nice facts about the modulus of a complex number.

If $\left|z\right| = 0$ then $z = 0$

$$\left|z_1\, z_2\right| = \left|z_1\right|\left|z_2\right| \qquad (14)$$

$$\left|\frac{z_1}{z_2}\right| = \frac{\left|z_1\right|}{\left|z_2\right|} \qquad (15)$$

Property (13) should make some sense to you. If the modulus is zero then $a^2 + b^2 = 0$, but the only way this can be zero is if both *a* and *b* are zero.

To verify (14) consider the following,

$$\begin{aligned}
\left|z_1\, z_2\right|^2 &= \left(z_1\, z_2\right)\overline{\left(z_1\, z_2\right)} && \text{using property (10)} \\
&= \left(z_1\, z_2\right)\left(\overline{z_1}\; \overline{z_2}\right) && \text{using property (4)} \\
&= z_1\overline{z_1}\; z_2\overline{z_2} && \text{rearranging terms} \\
&= \left|z_1\right|^2 \left|z_2\right|^2 && \text{using property (10) again (twice)}
\end{aligned}$$

So, from this we can see that

$$\left|z_1\, z_2\right|^2 = \left|z_1\right|^2 \left|z_2\right|^2$$

Finally, recall that we know that the modulus is always positive so take the square root of both sides to arrive at $|z_1 z_2| = |z_1||z_2|$

Property (15) can be verified using a similar argument.

Triangle Inequality and Variants

Properties (14) and (15) relate the modulus of a product/quotient of two complex numbers to the product/quotient of the modulus of the individual numbers. We now need to take a look at a similar relationship for sums of complex numbers. This relationship is called the **triangle inequality** and is,

$$|z_1 + z_2| \le |z_1| + |z_2| \qquad (16)$$

We'll also be able to use this to get a relationship for the difference of complex numbers.

The triangle inequality is actually fairly simple to prove so let's do that. We'll start with the left side squared and use (10) and (3) to rewrite it a little.

$$\left|z_1 + z_2\right|^2 = \left(z_1 + z_2\right)\left(\overline{z_1 + z_2}\right) = \left(z_1 + z_2\right)\left(\overline{z_1} + \overline{z_2}\right)$$

Now multiply out the right side to get,

$$\left|z_1 + z_2\right|^2 = z_1\,\overline{z_1} + z_1\,\overline{z_2} + z_2\,\overline{z_1} + z_2\,\overline{z_2} \qquad (17)$$

Next notice that,

$$\overline{z_2\overline{z_1}} = \overline{z_2}\,\overline{\overline{z_1}} = \overline{z_2}z_1$$

and so using (6), (8) and (11) we can write middle two terms of the right side of (17) as

$$z_1\,\overline{z_2} + z_2\,\overline{z_1} = z_1\,\overline{z_2} + \overline{z_1\,\overline{z_2}} = 2\,\mathrm{Re}\left(z_1\,\overline{z_2}\right) \le 2\left|z_1\,\overline{z_2}\right| = 2\left|z_1\right|\left|\overline{z_2}\right| = 2\left|z_1\right|\left|z_2\right|$$

Also use (10) on the first and fourth term in (17) to write them as,

$$z_1 \, \overline{z}_1 = \left| z_1 \right|^2 \qquad\qquad z_2 \, \overline{z}_2 = \left| z_2 \right|^2$$

With the rewrite on the middle two terms we can now write (17) as

$$\begin{aligned}
\left| z_1 + z_2 \right|^2 &= z_1 \, \overline{z}_1 + z_1 \, \overline{z}_2 + z_2 \, \overline{z}_1 + z_2 \, \overline{z}_2 \\
&= \left| z_1 \right|^2 + z_1 \, \overline{z}_2 + z_2 \, \overline{z}_1 + \left| z_1 \right|^2 \\
&\le \left| z_1 \right|^2 + 2\left| z_1 \right| \left| z_2 \right| + \left| z_1 \right|^2 \\
&= \left(\left| z_1 \right| + \left| z_2 \right| \right)^2
\end{aligned}$$

So, putting all this together gives,

$$\left| z_1 + z_2 \right|^2 \le \left(\left| z_1 \right| + \left| z_2 \right| \right)^2$$

Now, recalling that the modulus is always positive we can square root both sides and we’ll arrive at the triangle inequality.

$$\left| z_1 + z_2 \right| \le \left| z_1 \right| + \left| z_2 \right|$$

There are several variations of the triangle inequality that can all be easily derived.

Let's first start by assuming that $|z_1| \geq |z_2|$. This is not required for the derivation, but will help to get a more general version of what we're going to derive here. So, let's start with $|z_1|$ and do some work on it.

$$\begin{aligned} |z_1| &= |z_1 + z_2 - z_2| \\ &\leq |z_1 + z_2| + |-z_2| \qquad \text{Using triangle inequality} \\ &= |z_1 + z_2| + |z_2| \end{aligned}$$

Now, rewrite things a little and we get,

$$|z_1 + z_2| \geq |z_1| - |z_2| \geq 0 \quad (18)$$

If we now assume that $|z_1| \leq |z_2|$ we can go through a similar process as above except this time switch z_1 and z_2 and we get,

$$\left|z_1+z_2\right| \geq \left|z_2\right|-\left|z_1\right| = -\left(\left|z_1\right|-\left|z_2\right|\right) \geq 0 \quad (19)$$

Now, recalling the definition of absolute value we can combine (18) and (19) into the following variation of the triangle inequality.

$$\left|z_1+z_2\right| \geq \left|\left|z_1\right|-\left|z_2\right|\right| \qquad (20)$$

Also, if we replace z_2 with $-z_2$ in (16) and (20) we arrive at two more variations of the triangle inequality.

$$\left|z_1-z_2\right| \leq \left|z_1\right|+\left|z_2\right| \quad (21)$$

$$\left|z_1-z_2\right| \geq \left|\left|z_1\right|-\left|z_2\right|\right| \quad (22)$$

On occasion you'll see (22) called the **reverse triangle inequality**.

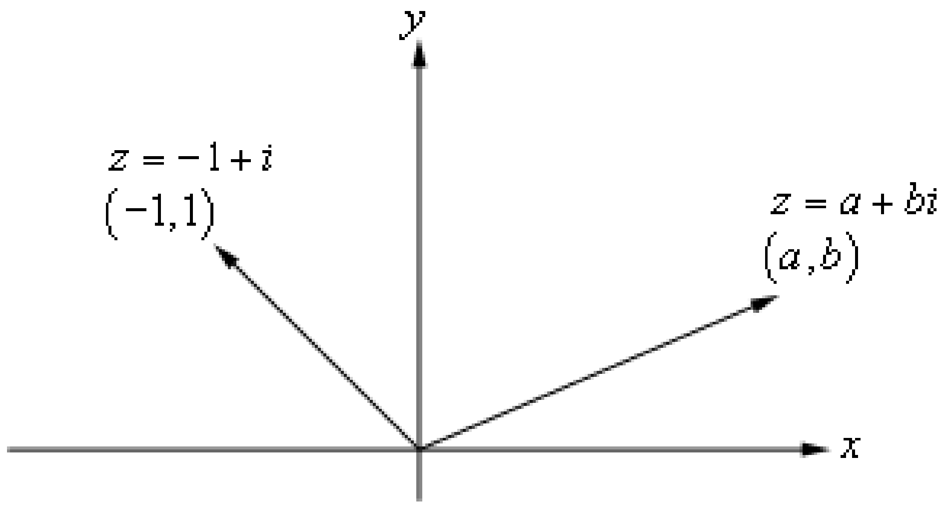
Polar & Exponential Form

Most people are familiar with complex numbers in the form $z = a + bi$, however there are some alternate forms that are useful at times. In this section we'll look at both of those as well as a couple of nice facts that arise from them.

Geometric Interpretation

Before we get into the alternate forms we should first take a very brief look at a natural geometric interpretation to a complex numbers since this will lead us into our first alternate form.

Consider the complex number $z = a + bi$. We can think of this complex number as either the point (a,b) in the standard Cartesian coordinate system or as the vector that starts at the origin and ends at the point (a,b). An example of this is shown in the figure below.



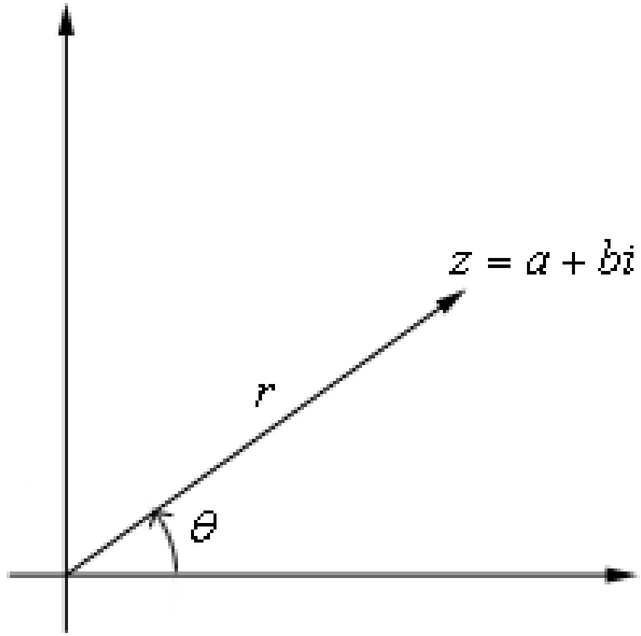
In this interpretation we call the *x*-axis the **real axis** and the *y*-axis the **imaginary axis**. We often call the *xy*-plane in this interpretation the **complex plane**.

Note as well that we can now get a geometric interpretation of the modulus. From the image above we can see that $|z| = \sqrt{a^2 + b^2}$ is nothing more than the length of the vector that we're using to represent the complex number

$z = a + bi$. This interpretation also tells us that the inequality $|z_1| < |z_2|$ means that z_1 is closer to the origin (in the complex plane) than z_2 is.

Polar Form

Let's now take a look at the first alternate form for a complex number. If we think of the non-zero complex number $z = a + bi$ as the point (a, b) in the *xy*-plane we also know that we can represent this point by the polar coordinates (r, θ), where *r* is the distance of the point from the origin and θ is the angle, in radians, from the positive *x*-axis to the ray connecting the origin to the point.



When working with complex numbers we assume that r is positive and that θ can be any of the possible (both positive and negative) angles that end at the ray. Note that this means that there are literally an infinite number of choices for θ.

We excluded $z = 0$ since θ is not defined for the point (0,0). We will therefore only consider the polar form of non-zero complex numbers.

We have the following *conversion* formulas for converting the polar coordinates (r,θ) into the corresponding Cartesian coordinates of the point, (a,b).

$$a = r\cos\theta \qquad\qquad b = r\sin\theta$$

If we substitute these into $z = a + bi$ and factor an *r* out we arrive at the ***polar form*** of the complex number,

$$z = r\left(\cos\theta + i\sin\theta\right) \qquad (1)$$

Note as well that we also have the following formula from polar coordinates relating *r* to *a* and *b*.

$$r = \sqrt{a^2 + b^2}$$

but, the right side is nothing more than the definition of the modulus and so

we see that, $r = |z|$ (2)

So, sometimes the polar form will be written as,

$$z = |z|\left(\cos\theta + i\sin\theta\right) \qquad (3)$$

The angle θ is called the **argument** of z and is denoted by, $\theta = \arg z$

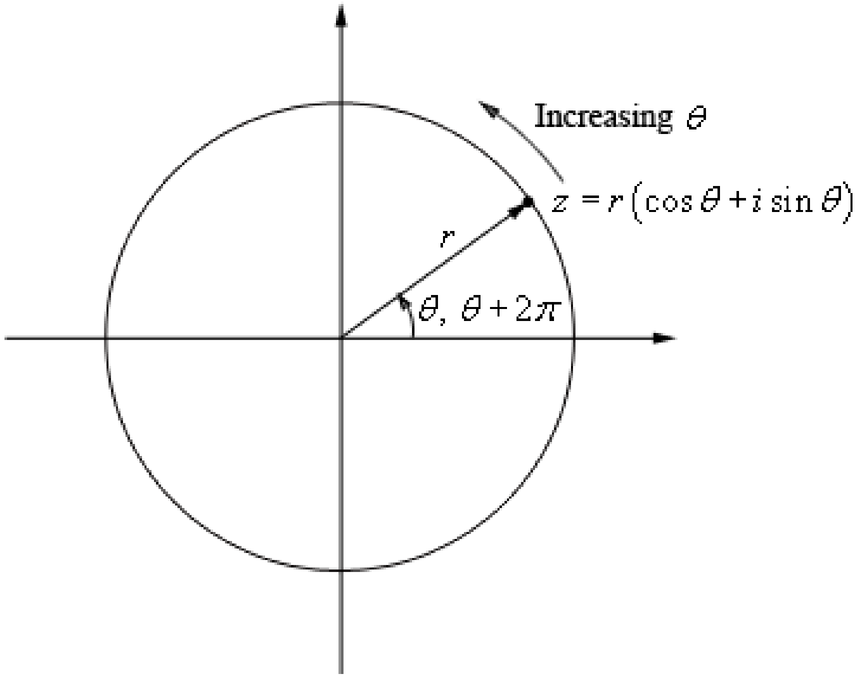
The argument of z can be any of the infinite possible values of θ each of which can be found by solving

$$\tan\theta = \frac{b}{a} \qquad (4)$$

and making sure that θ is in the correct quadrant.

Note as well that any two values of the argument will differ from each other by an integer multiple of 2π. This makes sense when you consider the following.

For a given complex number z pick any of the possible values of the argument, say θ. If you now increase the value of θ, which is really just increasing the angle that the point makes with the positive x-axis, you are rotating the point about the origin in a counter-clockwise manner. Since it takes 2π radians to make one complete revolution you will be back at your initial starting point when you reach $\theta + 2\pi$ and so have a new value of the argument. See the figure below.



If you keep increasing the angle you will again be back at the starting point when you reach $\theta + 4\pi$, which is again a new value of the argument. Continuing in this fashion we can see that every time we reach a new value of the argument we will simply be adding multiples of 2π onto the original value of the argument.

Likewise, if you start at θ and decrease the angle you will be rotating the point about the origin in a clockwise manner and will return to your original starting point when you reach $\theta - 2\pi$. Continuing in this fashion and we can again see that each new value of the argument will be found by subtracting a multiple of 2π from the original value of the argument.

So we can see that if θ_1 and θ_2 are two values of arg *z* then for some integer *k* we will have,

$$\theta_1 - \theta_2 = 2\pi k \qquad (5)$$

Note that we've also shown here that $z = r(\cos\theta + i\sin\theta)$ is a parametric representation for a circle of radius *r* centered at the origin and that it will trace out a complete circle in a counter-clockwise direction as the angle increases from θ to $\theta + 2\pi$.

The **principal value** of the argument (sometimes called the **principal argument**) is the unique value of the argument that is in the range

$-\pi < \arg z \le \pi$ and is denoted by $\operatorname{Arg} z$. Note that the inequalities at either end of the range tells that a negative real number will have a principal value of the argument of $\operatorname{Arg} z = \pi$.

Recalling that we noted above that any two values of the argument will differ from each other by a multiple of 2π leads us to the following fact.

$$\arg z = \operatorname{Arg} z + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots \tag{6}$$

We should probably do a couple of quick numerical examples at this point before we move on to look the second alternate form of a complex number.

Example 1 Write down the polar form of each of the following complex numbers.

(a) $z = -1 + i\sqrt{3}$ **(b)** $z = -9$ **(c)** $z = 12i$

Solution

(a) Let's first get *r*.

$$r = |z| = \sqrt{1+3} = 2$$

Now let's find the argument of *z*. This can be any angle that satisfies (4), but it's usually easiest to find the principal value so we'll find that one. The principal value of the argument will be the value of θ that is in the range $-\pi < \theta \le \pi$, satisfies,

$$\tan\theta = \frac{\sqrt{3}}{-1} \qquad \Rightarrow \qquad \theta = \tan^{-1}\left(-\sqrt{3}\right)$$

and is in the second quadrant since that is the location the complex number in the complex plane.

If you're using a calculator to find the value of this inverse tangent make sure that you understand that your calculator will only return values in the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and so you may get the incorrect value. Recall that if your calculator returns a value of θ_1 then the second value that will also satisfy the

equation will be $\theta_2 = \theta_1 + \pi$. So, if you're using a calculator be careful. You will need to compute both and the determine which falls into the correct quadrant to match the complex number we have because only one of them will be in the correct quadrant.

In our case the two values are,

$$\theta_1 = -\frac{\pi}{3} \qquad\qquad \theta_2 = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$$

The first one is in quadrant four and the second one is in quadrant two and so is the one that we're after. Therefore, the principal value of the argument is,

$$\operatorname{Arg} z = \frac{2\pi}{3}$$

and all possible values of the argument are then

$$\arg z = \frac{2\pi}{3} + 2\pi n \qquad\qquad n = 0, \pm 1, \pm 2, \ldots$$

Now, let's actually do what we were originally asked to do. Here is the polar form of $z = -1 + i\sqrt{3}$.

$$z = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$$

Now, for the sake of completeness we should acknowledge that there are many more equally valid polar forms for this complex number. To get any of the other forms we just need to compute a different value of the argument by picking *n*. Here are a couple of other possible polar forms.

$$z = 2\left(\cos\left(\frac{8\pi}{3}\right) + i\sin\left(\frac{8\pi}{3}\right)\right) \qquad n = 1$$

$$z = 2\left(\cos\left(-\frac{16\pi}{3}\right) + i\sin\left(-\frac{16\pi}{3}\right)\right) \qquad n = -3$$

(b) In this case we've already noted that the principal value of a negative real number is π so we don't need to compute that. For completeness sake here are all possible values of the argument of any negative number.

$$\arg z = \pi + 2\pi n = \pi\left(1 + 2n\right) \qquad n = 0, \pm 1, \pm 2, \ldots$$

Now, *r* is,

$r = |z| = \sqrt{81+0} = 9$

The polar form (using the principal value) is,

$z = 9\left(\cos(\pi) + i\sin(\pi)\right)$

Note that if we'd had a positive real number the principal value would be $\operatorname{Arg} z = 0$

(c) This another special case much like real numbers. If we were to use (4) to find the argument we would run into problems since the imaginary part is zero and this would give division by zero. However, all we need to do to get the argument is think about where this complex number is in the complex plane. In the complex plane purely imaginary numbers are either on the positive *y*-axis or the negative *y*-axis depending on the sign of the imaginary part.

For our case the imaginary part is positive and so this complex number will be on the positive y-axis. Therefore, the principal value and the general argument for this complex number is,

$$\operatorname{Arg} z = \frac{\pi}{2} \qquad \arg z = \frac{\pi}{2} + 2\pi n = \pi\left(\frac{1}{2} + 2n\right) \qquad n = 0, \pm 1, \pm 2, \ldots$$

Also, in this case r = 12 and so the polar form (again using the principal value) is,

$$z = 12\left(\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right)\right)$$

Exponential Form

Now that we've discussed the polar form of a complex number we can introduce the second alternate form of a complex number. First, we'll need Euler's formula,

$$\mathbf{e}^{i\theta} = \cos\theta + i\sin\theta \qquad (7)$$

With Euler's formula we can rewrite the polar form of a complex number into its ***exponential form*** as follows.

$$z = r\mathbf{e}^{i\theta}$$

where $\theta = \arg z$ and so we can see that, much like the polar form, there are an infinite number of possible exponential forms for a given complex number. Also, because any two arguments for a give complex number differ by an integer multiple of 2π we will sometimes write the exponential form as,

$$z = r\mathbf{e}^{i(\theta+2\pi n)} \qquad n = 0, \pm 1, \pm 2, \ldots$$

where θ is any value of the argument although it is more often than not the principal value of the argument.

To get the value of *r* we can either use (3) to write the exponential form or we can take a more direct approach. Let's take the direct approach. Take the modulus of both sides and then do a little simplification as follows,

$$|z| = \left|r\mathbf{e}^{i\theta}\right| = |r|\left|\mathbf{e}^{i\theta}\right| = |r||\cos\theta + i\sin\theta| = \sqrt{r^2+0}\,\sqrt{\cos^2\theta+\sin^2\theta} = r$$

and so we see that $r = |z|$.

Note as well that because we can consider $z = r(\cos\theta + i\sin\theta)$ as a parametric representation of a circle of radius *r* and the exponential form of a complex number is really another way of writing the polar form we can also consider $z = r\mathbf{e}^{i\theta}$ a parametric representation of a circle of radius *r*.

Now that we've got the exponential form of a complex number out of the way we can use this along with basic exponent properties to derive some nice facts about complex numbers and their arguments.

First, let's start with the non-zero complex number $z = r\mathbf{e}^{i\theta}$. In the arithmetic section we gave a fairly complex formula for the multiplicative inverse, however, with the exponential form of the complex number we can get a much nicer formula for the multiplicative inverse.

$$z^{-1} = \left(r\mathbf{e}^{i\theta}\right)^{-1} = r^{-1}\left(\mathbf{e}^{i\theta}\right)^{-1} = r^{-1}\mathbf{e}^{-i\theta} = \frac{1}{r}\mathbf{e}^{i(-\theta)}$$

Note that since r is a non-zero real number we know that $r^{-1} = \frac{1}{r}$. So, putting this together the exponential form of the multiplicative inverse is,

$$z^{-1} = \frac{1}{r}\mathbf{e}^{i(-\theta)} \qquad (8)$$

and the polar form of the multiplicative inverse is,

$$z^{-1} = \frac{1}{r}\left(\cos\left(-\theta\right) + i\sin\left(-\theta\right)\right) \qquad (9)$$

We can also get some nice formulas for the product or quotient of complex numbers. Given two complex numbers $z_1 = r_1\mathbf{e}^{i\theta_1}$ and $z_2 = r_2\mathbf{e}^{i\theta_2}$, where θ_1 is any value of $\arg z_1$ and θ_2 is any value of $\arg z_2$, we have

$$z_1 z_2 = \left(r_1\,\mathbf{e}^{i\theta_1}\right)\left(r_2\,\mathbf{e}^{i\theta_2}\right) = r_1\,r_2\mathbf{e}^{i\left(\theta_1+\theta_2\right)} \qquad (10)$$

$$\frac{z_1}{z_2} = \frac{r_1\,\mathbf{e}^{i\theta_1}}{r_2\,\mathbf{e}^{i\theta_2}} = \frac{r_1}{r_2}\mathbf{e}^{i\left(\theta_1-\theta_2\right)} \qquad (11)$$

The polar forms for both of these are,

$$z_1 z_2 = r_1\,r_2\left(\cos\left(\theta_1+\theta_2\right)+i\sin\left(\theta_1+\theta_2\right)\right) \qquad (12)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos\left(\theta_1-\theta_2\right)+i\sin\left(\theta_1-\theta_2\right)\right) \qquad (13)$$

We can also use (10) and (11) to get some nice facts about the arguments of a product and a quotient of complex numbers. Since θ_1 is any value of $\arg z_1$ and θ_2 is any value of $\arg z_2$ we can see that,

$$\arg\left(z_1\,z_2\right) = \arg z_1 + \arg z_2 \qquad (14)$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \qquad (15)$$

Note that (14) and (15) may or may not work if you use the principal value of the argument, Arg z. For example, consider $z_1 = i$ and $z_2 = -1$. In this case we have $z_1 z_2 = -i$ and the principal value of the argument for each is,

$$\operatorname{Arg}(i) = \frac{\pi}{2} \qquad \operatorname{Arg}(-1) = \pi \qquad \operatorname{Arg}(-i) = -\frac{\pi}{2}$$

However,

$$\operatorname{Arg}(i) + \operatorname{Arg}(-1) = \frac{3\pi}{2} \neq -\frac{\pi}{2}$$

and so (14) doesn't hold if we use the principal value of the argument. Note however, if we use,

$$\arg(i) = \frac{\pi}{2} \qquad \arg(-1) = \pi$$

then,

$$\arg(i) + \arg(-1) = \frac{3\pi}{2}$$

is valid since $\frac{3\pi}{2}$ is a possible argument for $-i$, it just isn't the principal value of the argument.

As an interesting side note, (15) actually does work for this example if we use the principal arguments. That won't always happen, but it does in this case so be careful!

We will close this section with a nice fact about the equality of two complex numbers that we will make heavy use of in the next section. Suppose that we have two complex numbers given by their exponential forms, $z_1 = r_1 \mathbf{e}^{i\theta_1}$ and $z_2 = r_2 \mathbf{e}^{i\theta_2}$. Also suppose that we know that $z_1 = z_2$. In this case we have,

$$r_1 \mathbf{e}^{i\theta_1} = r_2 \mathbf{e}^{i\theta_2}$$

Then we will have $z_1 = z_2$ if and only if,

$$r_1 = r_2 \qquad \text{and} \qquad \theta_2 = \theta_1 + 2\pi k \;\; \text{for some integer } k \;\; \left(i.e.\ k = 0, \pm 1, \pm 2, \ldots\right) \qquad (16)$$

Note that the phrase "if and only if" is a fancy mathematical phrase that means that if $z_1 = z_2$ is true then so is (16) and likewise, if (16) is true then we'll have $z_1 = z_2$.

This may seem like a silly fact, but we are going to use this in the next section to help us find the powers and roots of complex numbers.

Powers and Roots

In this section we're going to take a look at a really nice way of quickly computing integer powers and roots of complex numbers.

We'll start with integer powers of $z = r\mathbf{e}^{i\theta}$ since they are easy enough. If *n* is an integer then,

$$z^n = \left(r\mathbf{e}^{i\theta}\right)^n = r^n \mathbf{e}^{i\, n\theta} \qquad \textbf{(1)}$$

There really isn't too much to do with powers other than working a quick example.

Example 1 Compute $(3+3i)^5$.

Solution

Of course we could just do this by multiplying the number out, but this would be time consuming and prone to mistakes. Instead we can convert to exponential form and then use (1) to quickly get the answer.

Here is the exponential form of $3+3i$.

$$r = \sqrt{9+9} = 3\sqrt{2} \qquad \tan\theta = \frac{3}{3} \qquad \Rightarrow \qquad \operatorname{Arg} z = \frac{\pi}{4}$$

$$3+3i = 3\sqrt{2}\mathbf{e}^{i\frac{\pi}{4}}$$

Note that we used the principal value of the argument for the exponential form, although we didn't have to.

Now, use (1) to quickly do the computation.

$$
\begin{aligned}
(3+3i)^5 &= \left(3\sqrt{2}\right)^5 \mathbf{e}^{i\frac{5\pi}{4}} \\
&= 972\sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right)+i\sin\left(\frac{5\pi}{4}\right)\right) \\
&= 972\sqrt{2}\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right) \\
&= -972-972i
\end{aligned}
$$

So, there really isn't too much to integer powers of a complex number.
Note that if $r=1$ then we have,

$$z^n = \left(\mathbf{e}^{i\theta}\right)^n = \mathbf{e}^{i\,n\theta}$$

and if we take the last two terms and convert to polar form we arrive at a formula that is called **de Moivre's formula**.

$$\left(\cos\theta + i\sin\theta\right)^n = \cos\left(n\theta\right)+i\sin\left(n\theta\right) \qquad\qquad n=0,\pm1,\pm2,\ldots$$

We now need to move onto computing roots of complex numbers. We'll start this off "simple" by finding the **n^th^ roots of unity**. The n^th^ roots of unity for $n = 2, 3, \ldots$ are the distinct solutions to the equation,

$$z^n = 1$$

Clearly (hopefully) $z = 1$ is one of the solutions. We want to determine if there are any other solutions. To do this we will use the fact from the previous sections that states that $z_1 = z_2$ if and only if

$$r_1 = r_2 \qquad \text{and} \qquad \theta_2 = \theta_1 + 2\pi k \ \text{ for some integer } k \ \left(i.e.\ k = 0, \pm 1, \pm 2, \ldots\right)$$

So, let's start by converting both sides of the equation to complex form and then computing the power on the left side. Doing this gives,

$$\left(r\mathbf{e}^{i\theta}\right)^n = 1\mathbf{e}^{i(0)} \qquad \Rightarrow \qquad r^n \mathbf{e}^{i\,n\theta} = 1\mathbf{e}^{i(0)}$$

So, according to the fact these will be equal provided,

$$r^n = 1 \qquad\qquad n\theta = 0 + 2\pi k \quad k = 0, \pm 1, \pm 2, \ldots$$

Now, *r* is a positive integer (by assumption of the exponential/polar form) and so solving gives,

$$r = 1 \qquad\qquad \theta = \frac{2\pi k}{n} \qquad\qquad k = 0, \pm 1, \pm 2, \ldots$$

The solutions to the equation are then,

$$z = \exp\left(i\,\frac{2\pi k}{n} \right) \qquad\qquad k = 0, \pm 1, \pm 2, \ldots$$

Recall from our discussion on the polar form (and hence the exponential form) that these points will lie on the circle of radius *r*. So, our points will lie on the unit circle and they will be equally spaced on the unit circle at every $\frac{2\pi}{n}$ radians. Note this also tells us that there *n* distinct roots

corresponding to $k = 0, 1, 2, \ldots, n-1$ since we will get back to where we started once we reach $k = n$

Therefore there are n nth roots of unity and they are given by,

$$\exp\left(i\,\frac{2\pi k}{n}\right) = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right) \qquad k = 0, 1, 2, \ldots, n-1 \qquad (2)$$

There is a simpler notation that is often used to denote nth roots of unity. First define,

$$\omega_n = \exp\left(i\,\frac{2\pi}{n}\right) \qquad (3)$$

then the nth roots of unity are,

$$\omega_n^k = \left(\exp\left(i\,\frac{2\pi}{n}\right)\right)^k = \exp\left(i\,\frac{2\pi k}{n}\right) \qquad k = 0, 1, 2, \ldots n-1$$

Or, more simply the nth roots of unity are, $1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}$ (4)

where ω_n is defined in (3).

Example 2 Compute the nth roots of unity for n = 2, 3, and 4.

Solution

We'll start with n = 2.

$$\omega_2 = \exp\left(i\,\frac{2\pi}{2}\right) = \mathbf{e}^{i\pi}$$

This gives,

$$1 = 1 \qquad \text{and} \qquad \begin{aligned} \omega_2 &= \mathbf{e}^{i\pi} \\ &= \cos(\pi) + i\sin(\pi) \\ &= -1 \end{aligned}$$

So, for n = 2 we have -1, and 1 as the nth roots of unity. This should not be too surprising as all we were doing was solving the equation

$$z^2 = 1$$

and we all know that -1 and 1 are the two solutions.

While the result for $n = 2$ may not be that surprising that for $n = 3$ may be somewhat surprising. In this case we are really solving

$$z^3 = 1$$

and in the world of real numbers we know that the solution to this is $z = 1$. However, from the work above we know that there are 3 n^{th} roots of unity in this case. The problem here is that the remaining two are complex solutions and so are usually not thought about when solving for real solution to this equation which is generally what we wanted up to this point.

So, let's go ahead and find the n^{th} roots of unity for $n = 3$.

$$\omega_3 = \exp\left(i\,\frac{2\pi}{3}\right)$$

This gives,

$$1 = 1 \qquad \begin{aligned} \omega_3 &= \exp\left(i\,\frac{2\pi}{3}\right) \\ &= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) \\ &= -\frac{1}{2} + \frac{\sqrt{3}}{2} i \end{aligned} \qquad \begin{aligned} \omega_3^2 &= \exp\left(i\,\frac{4\pi}{3}\right) \\ &= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) \\ &= -\frac{1}{2} - \frac{\sqrt{3}}{2} i \end{aligned}$$

I'll leave it to you to check that if you cube the last two values you will in fact get 1.

Finally, let's go through $n = 4$. We'll do this one much quicker than the previous cases.

$$\omega_4 = \exp\left(i\,\frac{2\pi}{4}\right) = \exp\left(i\,\frac{\pi}{2}\right)$$

This gives,

$$1 = 1 \qquad \begin{aligned} \omega_4 &= \exp\left(i\,\frac{\pi}{2}\right) \\ &= i \end{aligned} \qquad \begin{aligned} \omega_4^2 &= \exp\left(i\,\pi\right) \\ &= -1 \end{aligned} \qquad \begin{aligned} \omega_4^3 &= \exp\left(i\,\frac{3\pi}{2}\right) \\ &= -i \end{aligned}$$

Now, let's move on to more general roots. First let's get some notation out of the way. We'll define $z_0^{1/n}$ to be any number that will satisfy the equation

$$z^n = z_0 \qquad (5)$$

To find the values of $z_0^{1/n}$ we'll need to solve this equation and we can do that in the same way that we found the n[th] roots of unity. So, if $r_0 = |z_0|$ and $\theta_0 = \arg z_0$ (note θ_0 can be any value of the argument, but we usually use the principal value) we have,

$$\left(r\mathbf{e}^{i\theta}\right)^n = r_0\,\mathbf{e}^{i\,\theta_0} \qquad \Rightarrow \qquad r^n\mathbf{e}^{i\,n\theta} = r_0\,\mathbf{e}^{i\,\theta_0}$$

So, this tells us that,

$$r = \sqrt[n]{r_0} \qquad \theta = \frac{\theta_0}{n} + \frac{2\pi k}{n} \qquad k = 0, \pm 1, \pm 2, \ldots$$

The distinct solutions to (5) are then,

$$a_k = \sqrt[n]{r_0}\exp\left(i\left(\frac{\theta_0}{n} + \frac{2\pi k}{n}\right)\right) \qquad k = 0, 1, 2, \ldots, n-1 \qquad (6)$$

So, we can see that just as there were n n[th] roots of unity there are also n n[th] roots of z_0.

Finally, we can again simplify the notation up a little. If *a* is any of the n$^{\text{th}}$ roots of z_0 then all the roots can be written as, $a, a\omega_n, a\omega_n^2, \ldots,, a\omega_n^{n-1}$ where ω_n is defined in (3).

Example 3 Compute all values of the following.

(a) $(2i)^{\frac{1}{2}}$ **(b)** $\left(\sqrt{3}-i\right)^{\frac{1}{3}}$

Solution

(a) The first thing to do is write down the exponential form of the complex number we're taking the root of.

$$2i = 2\exp\left(i\frac{\pi}{2}\right)$$

So, if we use $\theta_0 = \frac{\pi}{2}$ we can use (6) to write down the roots.

$$a_k = \sqrt{2}\exp\left(i\left(\frac{\pi}{4}+\pi k\right)\right) \qquad k = 0,1$$

Plugging in for k gives,

$$a_0 = \sqrt{2}\exp\left(i\frac{\pi}{4}\right) \qquad\qquad a_1 = \sqrt{2}\exp\left(i\left(\frac{5\pi}{4}\right)\right)$$

$$= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right) \qquad = \sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right)+i\sin\left(\frac{5\pi}{4}\right)\right)$$

$$= 1+i \qquad\qquad = -1-i$$

I'll leave it to you to check that if you square both of these will get $2i$.

(b) Here's the exponential form of the number, $\sqrt{3}-i = 2\exp\left(i\left(-\frac{\pi}{6}\right)\right)$

Using (6) the roots are,

$$a_k = \sqrt[3]{2}\exp\left(i\left(-\frac{\pi}{18}+\frac{2\pi k}{3}\right)\right) \quad k = 0,1,2$$

Plugging in for k gives,

$$a_0 = \sqrt[3]{2}\exp\left(i\left(-\frac{\pi}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(-\frac{\pi}{18}\right) + i\sin\left(-\frac{\pi}{18}\right)\right) = 1.24078 + 0.21878\,i$$

$$a_1 = \sqrt[3]{2}\exp\left(i\frac{11\pi}{18}\right) = \sqrt[3]{2}\left(\cos\left(\frac{11\pi}{18}\right) + i\sin\left(\frac{11\pi}{18}\right)\right) = -0.43092 + 1.18394\,i$$

$$a_2 = \sqrt[3]{2}\exp\left(i\frac{23\pi}{18}\right) = \sqrt[3]{2}\left(\cos\left(\frac{23\pi}{18}\right) + i\sin\left(\frac{23\pi}{18}\right)\right) = -0.80986 - 0.96516\,i$$

As with the previous part I'll leave it to you to check that if you cube each of these you will get $\sqrt{3} - i$.